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Phase Transition in the Nearest-Neighbor Continuum Potts Model

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In the present study we establish a phase transition in the nearest-neighbor continuum Potts model. The repulsion between particles of different type acts only on a nearest-neighbor graph, more precisely a subgraph of the Delaunay graph. This work is an adaptation of the Lebowitz and Lieb soft-core continuum Potts model.

KEY WORDS: Gibbs states; Delaunay triangulation; pairwise interaction; D.L.R. equations; local specifications; phase transition.

1. INTRODUCTION

The study of phase transitions holds an important place in statistical mechanics and in probability theory. Of course many results are known in the lattice case, but in the continuous setting, the situation is quite different and rather unsatisfactory. The case of the Widom–Rowlinson was solved in 1971 by Ruelle[(27)] using a version of the Peierls argument. Lebowitz and Lieb[(18)] extended his result by replacing the hard-core repulsion by a soft-core repulsion based on the distance between unlike particles. They require that the soft-core repulsion be sufficiently high or that the temperature be low enough. The extension of these results to continuum Potts models can be found in ref. 12, allowing global superstable interaction in a modern large-deviation spirit and in connection with percolation theory (see also ref. 8). The interest in percolation problems has grown rapidly during the last decades: see, for example, Meester and Roy[(22)] for continuum percolation, and also the references therein, Lyons and Peres[(20)] for percolation on

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trees and networks, and Häggström[16] for percolation on more general graphs. Another way to explore phase transitions is by studing liquid vapor phase transitions of the Van der Waals type based on mean-field equations coupled with the powerful Pirogov–Sinai theory as described in ref. 19 in a continuous setting for some potential of the Kac form.

In previous papers, we were interested in the nearest-neighbor Gibbs models introduced by Baddeley and Møller.[1] We have already proven the existence and unicity at small activity of stationary Gibbs states for such models in refs. 2–4.

The present paper gives a result of phase transition of the Widom–Rowlinson type. It is an attempt to replace the soft repulsion between several species of particles based on the distance (as introduced in refs. 29, 18, and 12) by another kind of soft repulsion mainly based on the structure of some graph. The most simple multi-type particle system would be the one with no more interaction than constant pairwise interspecies repulsion acting on the neighbors of the Delaunay graph. This particular case is really interesting since the interspecies repulsion is clearly independent of the distance between particles and consequently some possible percolation phenomenon, when it occurs, would be independent of the activity parameter z. Since a phase transition for a model of such kind is mainly related to the percolation, one could assert that if phase transition occurs for some particular value of z then it will occur for any other value of z. This behavior is really different from the one encountered for models with soft interspecies repulsion which depend only on the distance between particles. Let us note that this behavior may occur also for more general forms of interaction function than the constant one. Unfortunately, the proof of the existence of such a Gibbs point process seems as difficult as in the one-type point process with constant pairwise interaction on the Delaunay graph. This special case looks like the classical Poisson point process, but the quasilocality and relative compactness properties seem to be difficult to prove. It reminds one of problems described, for example, in ref. 28 and the references therein. One way among others (see refs. 3 and 2) to avoid the problem of existence of the multi-type particle system is to add some classical hard-core component interaction between particles of any species and to assume some kind of range property. If the hard-core distance δ is small enough and the range distance R is large enough, one may expect that this new model will exhibit a behavior very similar to the original one (corresponding to $\delta = 0$ and $R = +\infty$) discussed above.

For the proof of the main result, we used as in ref. 12 the random-cluster representation and adapted technical lemmas in our case. We also used a standard comparison argument as in refs. 16 and 12 in order to prove site and bond percolation in the dependent case.

We would like to point out some differences between our model and the classical soft-core Lebowitz and Lieb model. In the planar case, the Delaunay graph contains a number of edges proportional to the number of points, whereas this number is quadratic in the complete graph. Moreover, the neighborhood of a point in the Delaunay graph depends on the configuration of the point process, whereas in the classical case it depends only on the distance between points. So the activity parameter z does not play the same role in the two models because of the self-similarity property of the Delaunay graph. If z increases, the Delaunay neighborhood of a given point remains the same, whereas in the classical case it becomes strongly connected. Thus, in the classical case, the probability in the edge drawing mechanism can be chosen arbitrarily small. It is known (see refs. 6 and 16), at least for an ergodic hard-core or Poisson point process that the critical bond or site value on the Delaunay graph is not trivial. All these differences cause some difficulties for the proof of the existence of a phase transition (for example, Lemma 3.4 of ref. 12 does not work). The price to pay is to have an interspecies repulsion strong enough in our model to keep the percolation property and then a phase transition. We can even approach the critical value by simulation for the repulsion parameter of our model (see Fig. 5).

We may hope that nearest-neighbor continuum models are interesting for low temperature (not too low for a classical approach) as an alternative of standard models on regular networks, because they allow more degrees of freedom and may find application in crystallography. We may think of the rigidity and plasticity properties of glasses or the study of ferromagnetic fluids or liquid crystals (smectic A, C, nematic N). See, for example, refs. 9 and 13 and the references therein. On the other hand, recall that one example of the utilization of the Ising model is the study of the order–disorder transition of binary alloys or ionic crystals observed by Bragg diffraction of X rays. It is well known that Voronoi graphs and regions (rather called the Wigner–Seitz grid and the Brillouin zone within the framework of physics) hold a fundamental place for the understanding of electrical current, wave propagation, and phase transitions.

The paper is organized as follows. After giving some notations and preliminaries in Section 2, we introduce our model based on the Delaunay graph (Section 3). In Section 4 we recall the construction of the random-cluster representation and we give the main results of Georgii and Häggström, slightly adapted in our case. In Section 5 we give a result of percolation for the nearest-neighbor continuum random-cluster distribution following the lines of a proof proposed by Häggström,[16] which allows a phase transition for the nearest-neighbor continuum Potts model. Finally, we present simulations for our model using the Metropolis Hasting algorithm in Section 6.

2. NOTATIONS AND DEFINITIONS

Without loss of generality, this study takes place in $\mathbb{R}^2$, though all the results are still true in $\mathbb{R}^d$ ($d \geqslant 2$).

Let us first introduce some general notations. $|A|$ denotes the Lebesgue measure, when the set A is a bounded Borel set of $\mathbb{R}^2$, and the counting measure, when the set A is discrete.

For any bounded Borel set Λ and real $\delta > 0$,

$$\Lambda \ominus \delta = \bigcap_{y \in B(0,\delta)} \{z + y, z \in \Lambda\}$$

denotes the δ-minus-sampling of Λ. For two sets A and A' such that $A' \subset A$,

$$A \setminus A' = \{x \in A, x \notin A'\}.$$

Furthermore, $\mathbb{1}_A(\cdot)$ denotes the indicator function of the set A.

Let $B(x_0, x_1, x_2)$ be the open ball which contains x_0, x_1, x_2 in its topological boundary and $R(B(x_0, x_1, x_2))$ be the corresponding radius.

Let $\mathscr{X}_f$ and $\mathscr{X}$ be the sets of finite and locally finite subsets of $\mathbb{R}^2$. The set of configurations in a measurable set $\Lambda \subset \mathbb{R}^2$ will be denoted by $\mathscr{X}_\Lambda$. In fact, we consider tesselations in general position as in Møller,[24] which are of probability one for any stationary point process. More precisely:

(a) no three points lie on a straight line of $\mathbb{R}^2$, and

(b) no four points lie on the boundary of a circle.

We fix here an integer $q \geqslant 2$ which is the number of different types of a site. Let $\mathbf{X} = (X_1,..., X_q) \in \mathscr{X}^{(q)}$ be uniquely determined by the pair (X, σ), where $X = sp(\mathbf{X}) = X_1 \cup \cdots \cup X_q \in \mathscr{X}$ is the set of all occupied positions and $\sigma = \sigma(\mathbf{X})\colon \mathbf{X} \to \{1,..., q\}$ such that $\sigma(x) = l$ if $x \in X_l$ $(l = 1,..., q)$. The set $\mathscr{X}$ is equipped with the σ-algebra generated by the counting variable $|X \cap \Delta|$ for $X \in \mathscr{X}$ and for bounded measurables $\Delta \subset \mathbb{R}^2$. $\mathscr{X}^{(q)}$ is equipped with the product σ-algebra restricted to $\mathscr{X}^q$.

Definition 1. Let $X \in \mathscr{X}$. The Delaunay triangulation $\mathrm{Del}_3(X)$ of X is the unique one in which the interior of the circle circumscribed by every triangle of the triangulation does not contain any point of X in its interior.[24] The Delaunay graph, $\mathrm{Del}_2(X)$, is then defined by the set of edges of $\mathrm{Del}_3(X)$ (see Fig. 1).

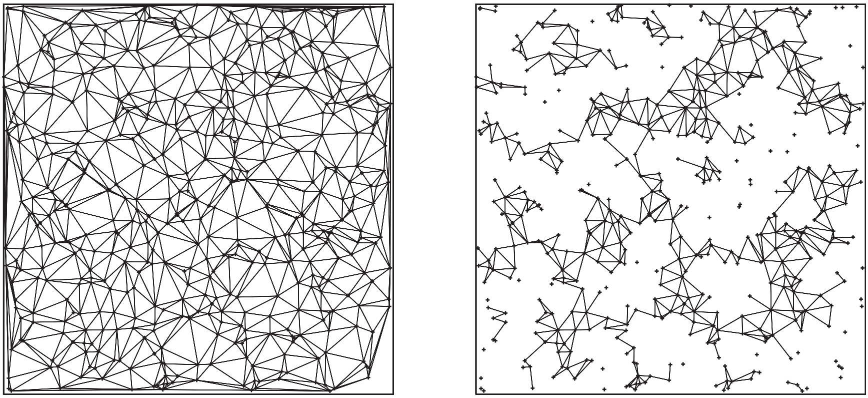

Fig. 1. The Delaunay graph for 500 sites at the left and the R_0-Delaunay graph for the same sites, $R_0 = 30$, at the right. The R_0-Delaunay graph is not necessarily a connected subgraph of the Delaunay graph.

Definition 2. The set of Delaunay triangles $S(x_0, x_1, x_2) \in \mathrm{Del}_3(X)$ such that

$$R(B(x_0, x_1, x_2)) > R_0$$

constitutes the Delaunay subgraphs $\mathrm{Del}_3^{R_0^+}(X)$ and $\mathrm{Del}_3^{R_0}(X) = \mathrm{Del}_3(X) \setminus \mathrm{Del}_3^{R_0^+}(X)$. Let $\mathrm{Del}_2^{R_0}(X)$ be the set of edges of $\mathrm{Del}_3^{R_0}(X)$ (see Fig. 1).

Definition 3. Let $X \in \mathscr{X}$ and x be a point of X. The Voronoi region $\mathrm{Vor}_X(x)$ associated with x is the set of points that are nearer to x than any of the other points of X. The Voronoi diagram is the set of all the Voronoi regions.

One can notice that all Voronoi regions are polygonal and convex. The Voronoi diagram is the dual orthogonal of the Delaunay triangulation. This characterization justifies that the Delaunay graph is a kind of "nearest-neighbor" graph.[1]

Let Λ be a bounded Borel set with a specific partition:

$$\Lambda = \bigcup_{k,l} \Delta_{k,l},$$

where

$$\Delta_{k,l} = \bigcup_{i,j=0}^{8} \Delta_{k,l}^{i,j}$$

and

$$\Delta_{k,l}^{i,j} = [9Lk + Li, 9Lk + L(i+1)] \times [9Ll + Lj, 9Ll + L(j+1)].$$

The set $\Delta_{k,l}$ is a square of size $9L$ and $\Delta_{k,l}^{i,j}$ is a "little" square of size L. This kind of partition was already introduced by Häggström in order to prove site and bond percolation on the Delaunay graph for almost all configurations of points of a stationary Poisson process.

Set $\epsilon = \frac{1-p_c^{\text{site}}(\mathbb{Z}^2)}{3}$, where $p_c^{\text{site}}(\mathbb{Z}^2)$ is the site percolation critical value for the square lattice.

3. PRESENTATION OF THE MODEL

Our model is a q-typed particle system in $\mathbb{R}^2$, with soft-core exclusion between particles of different color based on the R_0-Delaunay graph and hard-core pair interaction between all particles. Given $\mathbf{X} \in \mathcal{X}_f^q$, the finite energy is expressed as

$$V(\mathbf{X}) = V^{\psi}(X) + V^{\varphi}(\mathbf{X}), \tag{1}$$

where the first term does not depend on the color of the particles. We put a hard-core pair interaction between all particles:

$$V^{\psi}(X) = \sum_{\{x,y\} \subseteq X} \psi(\|x-y\|),$$

where

$$\psi(r) = \begin{cases} +\infty & \text{if} \quad r < \delta_0, \\ 0 & \text{otherwise.} \end{cases}$$

We keep this hard-core condition to control the local energy in Proposition 1 and for some technical reason in Lemma 1.

The second term in (1) describes the repulsion between particles of different type:

$$V^{\varphi}(\mathbf{X}) = \sum_{\substack{\{x,y\} \in \mathrm{Del}_2^{R_0}(X) \\ \sigma(x) \neq \sigma(y)}} \varphi(\|x-y\|).$$

For simplification we take

$$\varphi(r) = \begin{cases} A & \text{if} \quad r < 2R_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $A > 0$. With such potential φ, we can write:

$$V^{\varphi}(\mathbf{X}) = A|\{\{x, y\} \in \mathrm{Del}_2^{R_0}(X) : \sigma(x) \neq \sigma(y)\}|.$$

Furthermore, the hard-core distance δ_0 and the range R_0 satisfy the following assumption: there exists some constant L such that

$$0 < \delta_0 < \frac{L}{2} \quad \text{and} \quad \sqrt{2}\, L < 2R_0. \tag{2}$$

In particular, $\sqrt{2}\delta_0 < R_0$. The originality of our work is to show that the R_0-Delaunay repulsion (replacing the usual repulsion on the complete graph[18]) between particles of different type is strong enough to keep a phase transition.

Given a finite box $\Lambda \subset \mathbb{R}^2$ as in ref. 12, for the activity parameter z, the Gibbs distribution $Q_{\Lambda|\mathbf{Y}}$ on

$$\mathcal{R}_\Lambda = \{\mathbf{Y} \in \mathcal{X}_{\Lambda^c}^{(q)} : 0 < Z_{\Lambda|\mathbf{Y}} < \infty\},$$

where $Z_{\Lambda|\mathbf{Y}}$ is the partition function, is given by

$$Q_{\Lambda|\mathbf{Y}}(d\mathbf{X}) = Z_{\Lambda|\mathbf{Y}}^{-1} \exp(-H_{\Lambda|\mathbf{Y}}(\mathbf{X}))\, \pi_\Lambda^z(dX_1) \cdots \pi_\Lambda^z(dX_q), \tag{3}$$

where

$$H_{\Lambda|\mathbf{Y}}(\mathbf{X}) = \sum_{\substack{\{x, y\} \subseteq X \cup Y \\ \{x, y\} \cap \Lambda \neq \varnothing}} \psi(\|x-y\|) + \sum_{\substack{\{x, y\} \in \mathrm{Del}_2^{R_0}(X \cup Y) \\ \sigma(x) \neq \sigma(y) \\ \{x, y\} \cap \Lambda \neq \varnothing}} \varphi(\|x-y\|)$$

is the energy in Λ with the wired tempered boundary condition

$$\mathbf{Y} = (Y, \varnothing, \ldots, \varnothing)$$

of type 1 particles and π_Λ^z is the stationary Poisson point process on $\mathcal{X}_\Lambda$ with constant intensity z, i.e.,

$$\int f\, d\pi_\Lambda^z = \exp(-z\, |\Lambda|) \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\})\, dx_1 \cdots dx_n$$

for any bounded measurable function f on $\mathcal{X}_\Lambda$.

Definition 4. A Gibbs probability Q on $\mathcal{X}^{(q)}$ is a nearest-neighbor continuum Potts measure with activity z and interaction potentials φ and

ψ if the probability kernel $\mathbf{Y} \to Q_{\Lambda|\mathbf{Y}}$ from $\mathcal{X}_{\Lambda^c}^{(q)}$ to $\mathcal{X}_{\Lambda}^{(q)}$ is a version of the conditional distribution under Q of the configuration in Λ given the configuration outside Λ.

The existence of such a measure is easy to prove as an extension of ref. 3.

Proposition 1. There exists at least one nearest-neighbor continuum Potts measure.

Proof. The local energy required to insert a point $\mathbf{x}$ into a given configuration $\mathbf{X}$ is equal to:

$$\begin{aligned} E(\mathbf{x}, \mathbf{X}) &= V(\mathbf{X} \cup \{\mathbf{x}\}) - V(\mathbf{X}) \\ &= \sum_{y \in X} \psi(\|x-y\|) + \sum_{\substack{\{x,y\} \in \mathrm{Del}_2^{R_0}(X \cup \{x\}) \\ \sigma(x) \neq \sigma(y)}} \varphi(\|x-y\|) \\ &\quad - \sum_{\substack{\{y,z\} \in \mathrm{Del}_2^{R_0}(X) \\ \{y,z\} \notin \mathrm{Del}_2^{R_0}(X \cup \{x\}) \\ \sigma(y) \neq \sigma(z)}} \varphi(\|y-z\|). \end{aligned}$$

We can easily see that

$$E(\mathbf{x}, \mathbf{X}) = E(\mathbf{x}, \mathbf{X} \cap B(x, 2R_0)) \tag{4}$$

and because of the finite range property (Eq. (4)) and the hard-core condition, we have:

$$E(\mathbf{x}, \mathbf{X}) \geqslant -A \frac{4\pi R_0^2}{\pi \delta_0^2}. \tag{5}$$

Thus, the local energy has a range (Eq. (4)) and is bounded below (Eq. (5)). These two properties ensure that the set of Gibbs distributions with local specifications defined using $Q_{\Lambda|\mathbf{Y}}$ is not empty (see ref. 3). ∎

Now we are able to assert the main result of this article.

Theorem 1. If z and A are sufficiently large, there exist at least q distinct nearest-neighbor continuum Potts measures.

The rest of this paper is devoted to the proof of this theorem.

4. SOME RESULTS ON RANDOM-CLUSTER REPRESENTATION

For the convenience of the reader, we recall the main results given by Georgii and Häggström, which are slightly adapted to our framework (ref. 12, Section 2).

As in Georgii and Häggström,[12] we construct a probability measure **P** using the Fortuin–Kasteleyn representation:

- *Distribution of particle positions.*

$$P^{zq}_{\Lambda \mid Y}(dX) = Z^{-1}_{\Lambda \mid Y} \exp\left(-\sum_{\substack{\{x, y\} \subseteq X \cup Y \\ \{x, y\} \cap \Lambda \neq \varnothing}} \psi(x-y)\right) \pi^{zq}_{\Lambda \mid Y}(dX)$$

- *Type-picking mechanism.* For a fixed set of positions $X \in \mathscr{X}_{X \mid Y}$, we denote by $\lambda_{X, \Lambda}$ the distribution of

$$(\{x \in X : \tau_x = s\})_{1 \leqslant s \leqslant q} \in \mathscr{X}^{(q)},$$

where $(\tau_x)_{x \in X \cap \Lambda}$ are independent and uniformly distributed on the q colors, whereas $\tau_x = 1$ for $x \in X \setminus \Lambda = Y$ (wired boundary condition).

- *Edge drawing mechanism.* Let $\mu_{X, \Lambda}$ denote the distribution of the random edge configuration $\{e \in E_X : \eta_e = 1\}$, where $(\eta_e)_{e \in E_X}$ are independent $\{0, 1\}$-valued random variables with

$$prob(\eta_e = 1) = p_\Lambda(e) = \begin{cases} 1 - \exp(-A\, \mathbb{1}_{\mathrm{Del}_2^{R_0}(X)}(e)) & \text{if } \ e = \{x, y\} \in E_{\mathbb{R}^2} \setminus E_{\Lambda^c}, \\ \mathbb{1}_{\mathrm{Del}_2^{R_0}(X)}(e) & \text{if } \ e \in E_{\Lambda^c}, \end{cases}$$

where $E_{\mathbb{R}^2} = \{e = \{x, y\} \subset \mathbb{R}^2 : x \neq y\}$ is the set of all possible edges between pairs of points in $\mathbb{R}^2$ and $E_\Delta = \{e \in E_{\mathbb{R}^2} : e \subseteq \Delta\}$ is the set of all edges within Δ. Here we have slightly adapted the edge drawing mechanism to our model.

The probability measure $\mathbf{P}^{zq}_{\Lambda \mid Y}$ on $\mathscr{X}^{(q)} \times \mathscr{E}$ is then defined by:

$$\mathbf{P}^{zq}_{\Lambda \mid Y}(\mathbf{dX}, dE) = Z^{-1}_{\Lambda \mid Y} P^{zq}_{\Lambda \mid Y}(dX)\, \lambda_{X, \Lambda}(\mathbf{dX})\, \mu_{X, \Lambda}(dE),$$

where $\mathscr{E} = \{E \subset E_{\mathbb{R}^2} : E \text{ locally finite}\}$. Let

$$\Omega = \left\{(\mathbf{X}, E) \in \mathscr{X}^{(q)} \times \mathscr{E} : \sum_{\{x, y\} \in E} \sum_{s=1}^{q} \mathbb{1}_{X_s}(x)(1 - \mathbb{1}_{X_s}(y)) = 0\right\};$$

then the following conditional measure is the random-cluster measure:

$$\mathbf{P}_{\Omega} = \mathbf{P}^{zq}_{\Lambda\,|\,Y}(\cdot \mid \Omega),$$

where $\mathbf{P}^{zq}_{\Lambda\,|\,Y}(\Omega) > 0$.

Now we can recall the following results:

— Proposition 2.1 of ref. 12: if we disregard the edges of the random-cluster representation and look only at the particle positions and their type, then we obtain the *nearest-neighbor continuum Potts model*. The proof is completely general and does not depend on the underlying graph. The key of the proof is the form of the probability p_Λ of the edge drawing mechanism.

— Proposition 2.2 of ref. 12: if we disregard the type of particles of the random-cluster representation and look only at the particle positions and the edges, then we obtain the *continuum random-cluster distribution*:

$$C_{\Lambda\,|\,Y}(dX, dE) = Z^{-1}_{\Lambda\,|\,Y} q^{K_\Lambda(X,E)} P^{z}_{\Lambda\,|\,Y}(dX)\, \mu_{X,\Lambda}(dE), \tag{6}$$

where $K_\Lambda(X, E)$ is the number of connected components of (X, E) that are completely contained in Λ plus the infinite cluster outside Λ due to the wired boundary condition.

— Proposition 2.3 of ref. 12: we may express a relation between the number of particles of type 1 inside any box $\Delta \subset \Lambda$ and the percolation property of the random-cluster model, which is the key of the proof of the main result (Theorem 1):

Proposition 2. For any measurable $\Delta \subset \Lambda$,

$$\int (qN_{\Delta,1} - N_\Delta)\, dQ_{\Lambda\,|\,\mathrm{Y}} = (q-1) \int N_{\Delta \leftrightarrow \Lambda^c}\, dC_{\Lambda\,|\,Y},$$

where, for any box $\Delta \subset \mathbb{R}^2$,

$$N_\Delta(X) = |\{x \in X \cap \Delta\}|$$

is the random variable which represents the number of points in $X \cap \Delta$,

$$N_{\Delta,1}(X) = |\{x \in X \cap \Delta : \sigma(x) = 1\}|$$

is the random variable which represents the number of points of color 1 in $X \cap \Delta$,

$$N_{\Delta \leftrightarrow \Lambda^c}(X, E) = |\{x \in X \cap \Delta : x \text{ belongs to a cluster connected to } \Lambda^c \text{ in } (X, E \cap \mathrm{Del}_2^{R_0}(X))\}|$$

is the random variable which represents the number of points in $X \cap \Delta$ connected to any point in Λ^c in the random graph $E \cap \mathrm{Del}_2^{R_0}(X)$, and

$$\{\Delta \leftrightarrow \Lambda^c\} = \{N_{\Delta \leftrightarrow \Lambda^c} > 0\}$$

is the event that there exists at least a point in $X \cap \Delta$ and a point in Y_Λ^c connected in the random graph $E \cap \mathrm{Del}_2^{R_0}(X)$.

Remark 1. Because of the edge drawing mechanism, $\{\Delta \leftrightarrow \Lambda^c\}$ is also the event that there exists a point in $X \cap \Delta$ connected to infinity in the random graph $E \cap \mathrm{Del}_2^{R_0}(X)$.

One can write $C_{\Lambda|Y}$ as follows:

$$C_{\Lambda|Y}(dX, dE) = M_{\Lambda|Y}(dX)\, \mu_{X,\Lambda}^{(q)}(dE),$$

where

$$\mu_{X,\Lambda}^{(q)}(dE) = \frac{q^{K_\Lambda(X,E)}\mu_{X,\Lambda}(dE)}{\int q^{K_\Lambda(X,E)}\mu_{X,\Lambda}(dE)}$$

and the distribution of particle positions $M_{\Lambda|Y}$ is given by the marginal distribution $C_{\Lambda|Y}(\cdot, \mathscr{E})$. With such writing, it is easy to see that we have

$$\mu_{X,\Lambda}^{(q)} \succcurlyeq \tilde{\mu}_X, \tag{7}$$

where $\tilde{\mu}_X$ denotes the distribution of the random edge configuration $\{e \in E_X : \tilde{\eta}_e = 1\} \in \mathscr{E}$, where $(\tilde{\eta}_e)_{e \in E_X}$ are independent $\{0, 1\}$-valued random variables with

$$prob(\tilde{\eta}_e = 1) = \tilde{p} = \frac{1 - \exp(-A\, \mathbb{1}_{\mathrm{Del}_2^{R_0}(X)}(e))}{1 + (q-1)\exp(-A\, \mathbb{1}_{\mathrm{Del}_2^{R_0}(X)}(e))}.$$

For our proof of the phase transition, we will need here an additional type-picking mechanism. For a fixed set of positions $X \in \mathscr{X}^q$, we let $\tilde{\lambda}_X$ denote the distribution of

$$(\{x \in X : \tilde{\tau}_x = s\})_{1 \leqslant s \leqslant q} \in \mathscr{X}^{(q)},$$

where $(\tilde{\tau}_x)_{x \in X}$ are independent Bernoulli distributed:

$$prob(\tilde{\tau}_x = s) = \begin{cases} \tilde{p} & \text{if } \; s = 1, \\ 1 - \tilde{p} & \text{if } \; s \neq 1. \end{cases}$$

5. PERCOLATION IN THE R_0-DELAUNAY CONTINUUM RANDOM-CLUSTER MODEL

Now we establish percolation for the R_0-Delaunay continuum random-cluster distribution $C_{\Lambda|Y}$ for sufficiently large z and A and the boundary condition Y containing sufficiently many particles (Proposition 3). The proof is done using a standard comparison argument. Then the result of phase transition (Theorem 1) for the R_0-Delaunay continuum Potts model follows from Proposition 2. Let

$$N_{\Delta\leftrightarrow\Lambda^c}(\mathbf{X}) = |\{x \in X \cap \Delta : x \text{ belongs to a 1-cluster connected to } \Lambda^c \text{ in } (\mathbf{X}, \mathrm{Del}_2^{R_0}(X)\}|$$

be the random variable which represents the number of points in $X \cap \Delta$ connected to Λ^c by points of type 1 in $\mathrm{Del}_2^{R_0}(X)$ and

$$\{\boldsymbol{\Delta} \leftrightarrow \boldsymbol{\Lambda}^c\} = \{N_{\Delta\leftrightarrow\Lambda^c} > 0\}$$

the event that there exists an R_0-Delaunay path of particles of the same type between a particle of Δ and a particle of Λ^c. Let us define for each event A which is measurable in Λ:

$$\widetilde{C}^{\text{site}}_{\Lambda|\mathbf{Y}_{\Lambda^c}}(A) = \int M_{\Lambda|Y^c_\Lambda}(dX) \int \tilde{\lambda}_X(d\mathbf{X})\, \mathbb{1}_A(\mathbf{X}).$$

In order to control the measure $M_{\Lambda|Y}$, let us also define for $X \in \mathscr{X}_{\Lambda|Y}$:

$$h_\Lambda(X) = Z^{-1}_{\Lambda|Y} \int \mu_{X,\Lambda}(dE)\, q^{K_\Lambda(X,E)},$$

where $Z_{\Lambda|Y}$ is as in (6). Note that h_Λ is the Radon–Nikodym derivative of $M_{\Lambda|Y}$ with respect to $P^z_{\Lambda|Y}$.

Lemma 1. $\exists \alpha > 0$ such that $\forall \Lambda \subset \mathbb{R}^2$, $X \in \mathscr{R}_\Lambda$, and $x \in \Lambda \setminus X$,

$$\frac{h_\Lambda(X \cup \{x\})}{h_\Lambda(X)} \geqslant \alpha > 0.$$

Proof. Let us denote:

$$E^{\text{ext}}_{x|X} = \mathrm{Del}_2^{R_0}(X) \cap \mathrm{Del}_2^{R_0}(X \cup \{x\}),$$
$$E^{+}_{x|X} = \mathrm{Del}_2^{R_0}(X \cup \{x\}) \setminus \mathrm{Del}_2^{R_0}(X) = \{\{x, y\} \in \mathrm{Del}_2^{R_0}(X \cup \{x\})\},$$
$$E^{-}_{x|X} = \mathrm{Del}_2^{R_0}(X) \setminus \mathrm{Del}_2^{R_0}(X \cup \{x\}) = \mathrm{Del}_2^{R_0}(X) \setminus E^{+}_{x|X}.$$

Furthermore,

- $\mu^{\text{ext}}_{x|X}$ is the edge drawing mechanism on the random edges $E^{\text{ext}}_{x|X}$;
- $\mu^{+}_{x|X}$ is the edge drawing mechanism on the random edges $E^{+}_{x|X}$;
- $\mu^{+}_{x|X}$ is the edge drawing mechanism on the random edges $E^{-}_{x|X}$.

$$\frac{h_\Lambda(X \cup \{x\})}{h_\Lambda(X)} = \frac{\int \mu_{X \cup \{x\}, \Lambda}(dE)\, q^{K(X \cup \{x\}, E)}}{\int \mu_{X, \Lambda}(dE)\, q^{K(X, E)}}$$

$$= \frac{\int \mu^{\text{ext}}_{x|X}(dE_1)\, q^{K(X, E_1)} \int \mu^{+}_{x|X}(dE_2)\, q^{K(X \cup \{x\}, E_1 \cup E_2) - K(X, E_1)}}{\int \mu^{\text{ext}}_{x|X}(dE_1)\, q^{K(X, E_1)} \int \mu^{-}_{x|X}(dE_2)\, q^{K(X, E_1 \cup E_2) - K(X, E_1)}}.$$

But, because of the finite range condition on φ and the hard-core condition on ψ, we have

$$K(X \cup \{x\}, E_1 \cup E_2) - K(X, E_1) \geqslant -\frac{4\pi R_0^2}{\pi \delta_0^2}.$$

Moreover,

$$K(X, E_1 \cup E_2) - K(X, E_1) \leqslant 0.$$

Thus,

$$\frac{h_\Lambda(X \cup \{x\})}{h_\Lambda(X)} \geqslant q^{-\frac{4R_0^2}{\delta_0^2}} = \alpha > 0. \quad \blacksquare$$

Lemma 2. For all cells $\nabla = \Delta^{i,j}_{k,l} \subset \Lambda$ and any $Y_{\nabla^c} \in \mathscr{R}_\nabla$, we have

$$M_{\Lambda, \nabla | Y_{\nabla^c}}(|X \cap \nabla| \geqslant 1) > 1 - \frac{\epsilon}{81}$$

for large enough z.

Proof.

$$\frac{M_{\Lambda, \nabla | Y_{\nabla^c}}(|X \cap \nabla| = 1)}{M_{\Lambda, \nabla | Y_{\nabla^c}}(|X \cap \nabla| = 0)} = z \int_\nabla \exp(-H^{\psi}_{\nabla | Y_{\nabla^c}}(\{x\}))\, \frac{h_\Lambda(Y_{\nabla^c} \cup \{x\})}{h_\Lambda(Y_{\nabla^c})}\, dx.$$

From Lemma 1 we have

$$\frac{M_{\Lambda, \nabla | Y_{\nabla^c}}(|X \cap \nabla| = 1)}{M_{\Lambda, \nabla | Y_{\nabla^c}}(|X \cap \nabla| = 0)} \geqslant z \int_\nabla \alpha \exp(-H^{\psi}_{\nabla | Y_{\nabla^c}}(\{x\}))\, dx$$

$$\geqslant \alpha z\, |\nabla \ominus \delta_0|.$$

Thus,

$$M_{\Lambda,\nabla\,|\,Y_{\nabla^c}}(|X\cap\nabla|=0)\leqslant\frac{1}{\alpha z\,|\nabla\ominus\delta_0|}$$

and

$$\begin{aligned}M_{\Lambda,\nabla\,|\,Y_{\nabla^c}}(|X\cap\nabla|\geqslant 1)&>1-\frac{1}{\alpha z\,|\nabla\ominus\delta_0|}\\&>1-\frac{\epsilon}{81}\end{aligned}$$

for z large enough ($|\nabla\ominus\delta_0|>0$ because $\delta_0<\frac{L}{2}$). ∎

The following is an adaptation of the proof of Proposition 7.3 of ref. 16 to our context.

Lemma 3. Let z and Λ be large enough. Then

$$\tilde{C}^{\text{site}}_{\Lambda\,|\,\mathrm{Y}_{\Lambda^c}}(\{\Delta\leftrightarrow\Lambda^c\})\geqslant\xi>0$$

for any $\Delta=\Delta_{k,l}\subset\Lambda$.

Proof. We proceed as in ref. 16. Let $\nabla=\Delta^{i,j}_{k,l}$ for some i and j in $\{0,\ldots,8\}$. By Lemma 2, we can take z large enough such that

$$\forall Y_{\nabla^c}\in\mathscr{R}_{\nabla},\qquad M_{\Lambda,\nabla\,|\,Y_{\nabla^c}}(|X\cap\nabla|=0)\leqslant\frac{\epsilon}{81},$$

which implies

$$\forall Y_{\Delta^c}\in\mathscr{R}_{\Delta},\qquad M_{\Lambda,\Delta\,|\,Y_{\Delta^c}}(|X\cap\nabla|=0)\leqslant\frac{\epsilon}{81}.$$

Then, for

$$A_{k,l}=\bigcap_{i,j=0}^{8}\,(|X\cap\Delta^{i,j}_{k,l}|\geqslant 1),$$

$$M_{\Lambda,\Delta\,|\,Y_{\Delta^c}}(A_{k,l})\geqslant 1-\sum_{i,j=0}^{8}M_{\Lambda,\Delta\,|\,Y_{\Delta^c}}(|X\cap\Delta^{i,j}_{k,l}|=0)>1-\epsilon>p_c^{\text{site}}(\mathbb{Z}^2).$$

Let $C_{k,l}$ be the following event:

$$C_{k,l} = \{\mathbf{X}\colon \forall X \in A_{k,l},\ \forall x \in X \cap \Delta_{k,l}\ ,\ \sigma(x) = 1\}.$$

Let Λ be large enough such that

$$\tilde{p} \geqslant (1-\epsilon)^{\frac{1}{81M}},$$

where

$$M = \left[\frac{L^2}{\pi\delta_0^2}\right] + 1$$

and $[\,\cdot\,]$ denotes the integer part. We have

$$\begin{aligned}
\widetilde{C}^{\text{site}}_{\Delta_{k,l} \mid \mathbf{Y}_{\Delta^c_{k,l}}}(C_{k,l}) &= \int M_{\Lambda, \Delta_{k,l} \mid Y_{\Delta^c_{k,l}}}(dX)\, \mathbb{1}_{A_{k,l}}(X)\, \tilde{p}^{|X \cap \Delta_{k,l}|} \\
&\geqslant (1-\epsilon)\, \tilde{p}^{81M} \geqslant (1-\epsilon)^2 \\
&> 1 - 2\epsilon > p_c^{\text{site}}(\mathbb{Z}^2). \qquad (8)
\end{aligned}$$

Then, by a result of percolation theory, there exists a path of boxes $\Delta_{i,j}$ such that $C_{i,j}$ occurs, from any $\Delta_{k,l} \subset \Lambda$ to Λ^c.

Assume that $A_{k,l}$ and $A_{k+1,l}$ occur simultaneously. Let us define the "central band" of $\Delta_{k,l} \cup \Delta_{k+1,l}$:

$$CB_{k:k+1,l} = \left(\bigcup_{i=0}^{4} \Delta_{k,l}^{4+i,4}\right) \cup \left(\bigcup_{i=0}^{4} \Delta_{k+1,l}^{i,4}\right).$$

We keep in all the squares $\Delta_{k,l}$ and $\Delta_{k+1,l}$ the points

$$H = \{x \in X \cap (\Delta_{k,l} \cup \Delta_{k+1,l}) : \operatorname{Vor}_X(x) \cap CB_{k:k+1,l} \neq \emptyset\}.$$

All the edges of the restriction of the graph $\operatorname{Del}_2^{R_0}(X)$ to H are smaller than $\sqrt{2}L$ because all the little squares $\Delta_{k,l}^{i,j}$, $\Delta_{k+1,l}^{i,j}$, $i, j = 0,\ldots, 8$, contain at least 1 point and the circles circumscribed to the Delaunay triangles are empty. Thus the restriction of the graph $\operatorname{Del}_2^{R_0}(X)$ to H is equal to the restriction of the graph $\operatorname{Del}_2(X)$ to H. Since the Voronoi polygons form a connected covering of the "central band," we are able to connect any point of $\Delta_{k,l}^{4,4}$ to any point of $\Delta_{k+1,l}^{4,4}$ in the graph $\operatorname{Del}_2^{R_0}(X)$ inside $\Delta_{k,l} \cup \Delta_{k+1,l}$.

Thus, using (8), we have

$$\widetilde{C}^{\text{site}}_{\Lambda \mid \mathbf{Y}_{\Lambda^c}}(\{\boldsymbol{\Delta} \leftrightarrow \boldsymbol{\Lambda}^c\}) \geqslant \xi > 0. \quad \blacksquare$$

Proposition 3. Let z and A be large enough. Then there exists $\xi > 0$ such that

$$\int dC_{\Lambda \mid Y_{\Lambda^c}} N_{\Delta \leftrightarrow \Lambda^c} \geqslant \xi > 0$$

for any $\Delta = \Delta_{k,l} \subset \Lambda$.

Proof.

$$\int C_{\Lambda \mid Y_{\Lambda^c}}(dX, dE)\, N_{\Delta \leftrightarrow \Lambda^c}(X, E) = \int M_{\Lambda \mid Y_{\Lambda^c}}(dX) \int \mu^{(q)}_{X, \Lambda}(dE)\, N_{\Delta \leftrightarrow \Lambda^c}(X, E) \tag{9}$$

$$\geqslant \int M_{\Lambda \mid Y_{\Lambda^c}}(dX) \int \tilde{\mu}_X(dE)\, N_{\Delta \leftrightarrow \Lambda^c}(X, E) \tag{10}$$

$$\geqslant \int M_{\Lambda \mid Y_{\Lambda^c}}(dX) \int \tilde{\mu}_X(dE)\, \mathbb{1}_{\{\Delta \leftrightarrow \Lambda^c\}}(X, E) \tag{11}$$

$$\geqslant \int M_{\Lambda \mid Y_{\Lambda^c}}(dX) \int \tilde{\lambda}_X(d\mathbf{X})\, \mathbb{1}_{\{\boldsymbol{\Delta} \leftrightarrow \boldsymbol{\Lambda}^c\}}(\mathbf{X}) \tag{12}$$

$$= \tilde{C}^{\text{site}}_{\Lambda \mid \mathbf{Y}_{\Lambda^c}}(\{\boldsymbol{\Delta} \leftrightarrow \boldsymbol{\Lambda}^c\}) \geqslant \xi > 0. \tag{13}$$

The inequality (10) is due to the stochastic domination of $\tilde{\mu}_X$ by $\mu^{(q)}_{X, \Lambda}$ (see Eq. (7)). As site percolation implies bond percolation, we have (12). Lemma 3 gives (13). ∎

Proof of Theorem 1. By Propositions 2 and 3 we have

$$\int dQ_{\Lambda \mid \mathbf{Y}^{(k)}}(qN_{\Delta, k} - N_\Delta) \geqslant (q-1)\, \epsilon$$

for $k = 1, \ldots, q$, where

$$\mathbf{Y}^{(k)} = (\varnothing, \ldots, \varnothing, Y, \varnothing, \ldots, \varnothing)$$

is a tempered boundary condition of type-k particles.

$$N_{\Delta, k}(X) = |\{x \in X \cap \Delta : \sigma(x) = k\}|$$

is the random variable which represents the number of sites of type k in $X \cap \Delta$. The existence with such a boundary condition for the nearest-neighbor continuum Potts measure is established in Proposition 1. Thus, Theorem 1 follows by the classical argument giving at least q distinct measures with distinct intensity of colors. ∎

Remark 2. Using similar arguments as in Proposition 3, the random-cluster distribution can be upper bounded by the so-called random edge model of hard-core particles on the Delaunay graph. It is known (see ref. 6), at least for an ergodic hard-core point process, that the critical bond or site value p_c^b, p_c^s on the Delaunay graph is bounded below. Thus, if A is small enough such that the edge probability becomes smaller than p_c^b, then there is no percolation for the random-cluster distribution and of course no phase transition of this type (with distinct intensity of colors, see Fig. 4).

Remark 3. We may ask if the q nearest-neighbor continuum Potts measures are ergodics and if a variational formula relating pressure, entropy, and internal energy can be obtained. One way is to try to apply the large-deviation theory based on empirical measures developed in refs. 11 and 12 for the model given in this paper and more generally for the nearest-neighbor models given in ref. 3.

Remark 4. An interesting question is to know on what Delaunay subgraph the repulsion is strong enough to maintain a phase transition. In terms of percolation, the question is to know if bond percolation is maintained in subgraphs of this type. Reference 6 gives a positive answer for the Gabriel graph. Furthermore, as the critical value is trivial and equal to 1 for the minimum spanning forest a negative answer should be given.

Remark 5. For the k-nearest-neighbor graph, we can see, following the lines of this article, that for k sufficiently large, the repulsion is strong enough to maintain a phase transition because of the result of percolation given in ref. 17.

6. SOME SIMULATIONS

Ripley[26] used Preston's spatial birth and death processes[25] to simulate spatial patterns in the fixed number case. The simulation procedure used here is a direct adaptation of the Geyer and Møller proposal[15, 14] to the R_0-Delaunay energy. The algorithm is the following (*rand*() designates a random variable uniformly distributed in $[0, 1]$):

— Initialize with a given permissible configuration. Here, the initialization is done with the empty configuration in the square $\Lambda = [0; 700]^2 \ominus 100$ and the *fixed wired boundary configuration* $\mathbf{Y}_{[0;\,700]^2 \setminus \Lambda}$ in the exterior of Λ where all the sites have the type 1 (see Figs. 2–4);

— Repeat a large number of times the following steps:

- $p = rand()$;
- if $(p \leqslant \frac{1}{2})$ then
  - choose x uniformly in X_Λ;
  - if $rand() \leqslant \min\{1, \frac{|X_\Lambda|}{z\,|\Lambda|}\, e^{E(\mathbf{x}, \mathbf{X}_\Lambda \setminus \{\mathbf{x}\} \cup \mathbf{Y}_{[0;\,700]^2 \setminus \Lambda})}\}$, then $\mathbf{X}_\Lambda \leftarrow \mathbf{X}_\Lambda \setminus \{\mathbf{x}\}$;

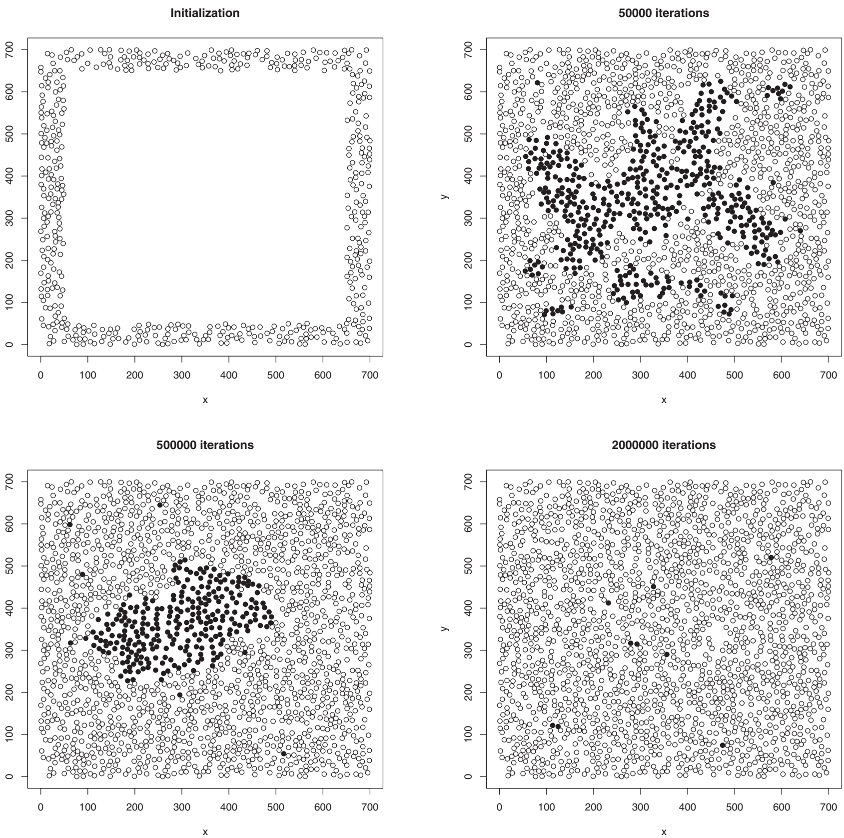

Fig. 2. Simulation of the Delaunay continuum Potts models with $z = 0.04$, $A = 1$, $R_0 = 30$, and $\delta_0 = 10$ in the square $[0, 700] \times [0, 700]$.

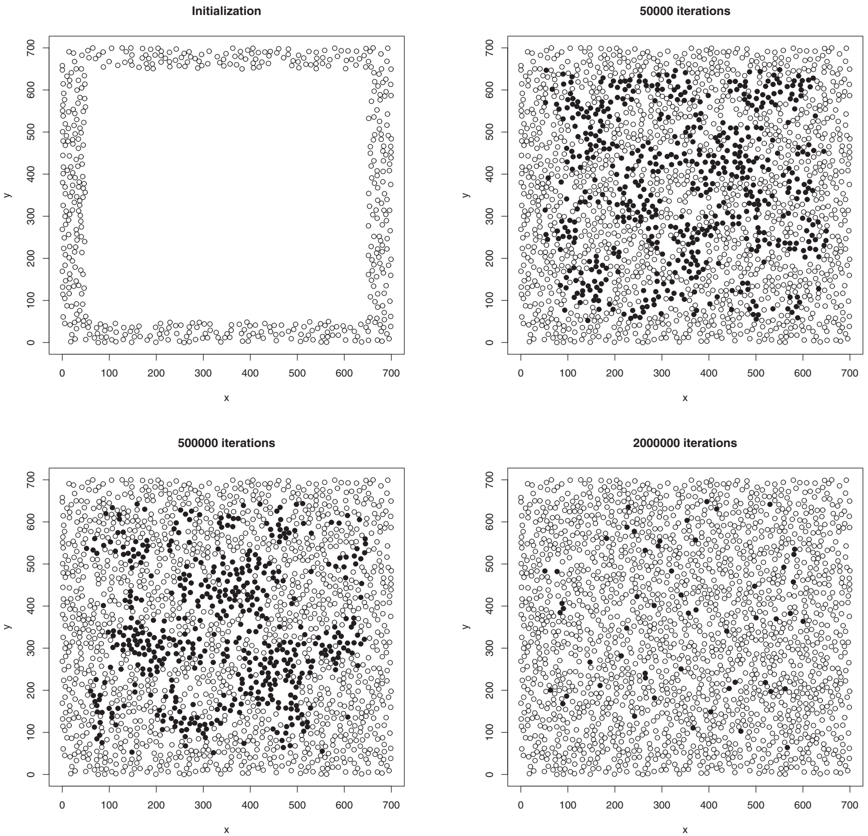

Fig. 3. Simulation of the Delaunay continuum Potts models with $z=0.04$, $A=0.7$, $R_0=30$, and $\delta_0=10$ in the square $[0,700]\times[0,700]$.

- if $(p>\frac{1}{2})$, then
  - choose x uniformly in Λ,
  - set $\sigma(x)=1$ with probability $\frac{1}{2}$,
  - if $rand()\leqslant\min\{1,\frac{z\,|\Lambda|}{|X_\Lambda|+1}\,e^{-E(\mathbf{x},\,\mathbf{X}_\Lambda\cup\mathbf{Y}_{[0;\,700]^2\setminus\Lambda})}\}$, then $\mathbf{X}_\Lambda\leftarrow\mathbf{X}_\Lambda\cup\{\mathbf{x}\}$.

The previous algorithm simulates a Markov chain which is Harris recurrent and geometrically ergodic with invariant Gibbs distributions. Indeed, as we have seen before, the local energy is lower bounded because of the range condition between particles of different type and the hard-core condition between all the particles (see Eq. (5)). One can note that there exist other dynamics to compute a realization of our model, but this is not the purpose of this article (see, for example, refs. 23, 10, and 21).

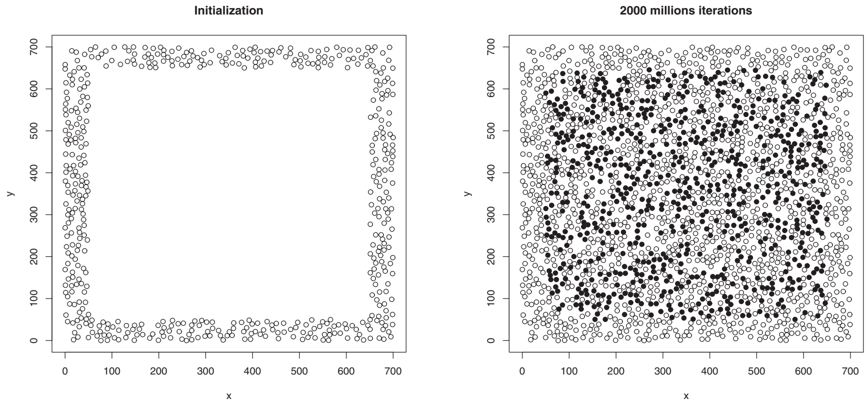

Fig. 4. Simulation of the Delaunay continuum Potts models with $z = 0.04$, $A = 0.1$, $R_0 = 30$, and $\delta_0 = 10$ in the square $[0, 700] \times [0, 700]$. A is too small to observe a phase transition.

An incremental approach is adopted (deletion and insertion[7]) in order to compute the Delaunay triangulation and to produce birth and death simulations. Furthermore, due to a Markov property,[5] the computation of $E(\mathbf{x}, \mathbf{X})$ is local. The Markov local property combined with the incremental approach leads to a great reduction of the execution time.

As we see in Figs. 2 and 3, after 2 million iterations almost all the sites of $\mathbf{X}_\Lambda$ have the same type as the exterior configuration $\mathbf{Y}_{[0;\,700]^2 \setminus \Lambda}$. Thus the boundary condition $\mathbf{Y}_{[0;\,700]^2 \setminus \Lambda}$ influences the type of sites of $\mathbf{X}_\Lambda$. This is a practical characterization of a phase-transition phenomenon for the

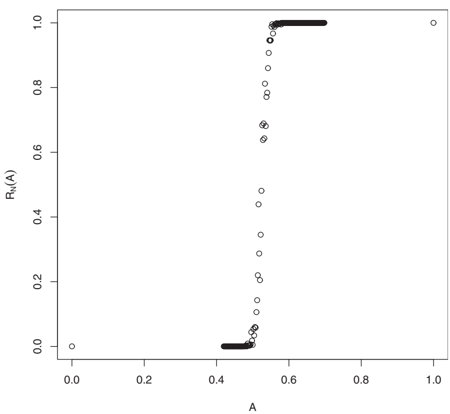

Fig. 5. R_0-Delaunay random-cluster percolation for $z = 0.04$, $R_0 = 30$, and $\delta_0 = 5$ in the $[0, 700] \times [0, 700]$ square. Experimentally, a critical value for the parameter A is approximately equal to 0.54.

R_0-Delaunay continuum Potts model. If A is too small, we do not observe this behavior as in Fig. 4 after 2000 million iterations.

In Fig. 5, we determine experimentally a critical value for the parameter A of our model. We determine a function $R_N(A)$ of the Delaunay random-cluster model in the following way. We carry out $N = 1000$ Monte Carlo (MC) runs for a given parameter A. If we find a percolating cluster from left to right in each of M runs, then $R_N(A) = \frac{M}{N}$. We repeat this process for different values of A with increment 0.002. Then we plot $R_N(A)$ against A at discrete values of A. Experimentally, the percolation threshold $p_c(z, R_0, \delta_0)$ is the value of A at which $R_N(A)$ becomes $\frac{1}{2}$. To give an idea of $p_c(z, R_0, \delta_0)$, we obtain approximately $p_c(0.04, 30, 10) \approx 0.54$ (Fig. 5).

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